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Global periodicity: An inverse problem

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Abstract

Given a positive integer p, we develop a method to construct difference equations of order greater than or equal to 2 such that all solutions of which are periodic of the same period p. The method of construction is based on a class of symmetric functions that we call "isovertible" functions.

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1. Introduction

Let $k \geq 2$ be a positive integer and consider the kth order difference equation (or recurrence relation):

$$y_{n+k} = f(y_n, \ldots, y_{n+k-1}), \quad n = 0, 1, 2, \ldots \tag{1}$$

where $f : D^k \to D$ is a continuous function and D is an interval of real numbers.

Given a positive integer $p \geq k$ and initial conditions $y_0, \ldots, y_{k-1} \in D$, the unique solution $\{y_n\}_{n=0}^{\infty}$ is said to be *periodic of period* p if $y_{n+p} = y_n$ for $n = 0, 1, 2, \ldots$. The least such integer is called the *fundamental or prime period* [6]. Periodicity (with the same period) of all solutions of Eq. (1) for certain classes of functions has been investigated in [1,4,5]. Studies of such problem in a more general setting have been done in [2,3,7,8].

Still periodicity (with the same period) of all solutions of Eq. (1) is our main concern in this research. However, our approach is different. It is more or less an inverse way of thinking. To be more precise:

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when the function f is given, a direct problem is to see if there exists a period p such that all solutions of Eq. (1) are periodic of period p. An inverse problem is: given a period p, can we find function(s) such that all solutions of Eq. (1) are periodic of this specified period p. It is the last problem that we tackle in this note.

2. Preliminaries

For ease of reference and convenience of the reader, we present the following definitions.

Definition 2.1. A function $F(x_1, \ldots, x_k)$ is said to be symmetric if it is invariant under any permutation of $(x_1, \ldots, x_k)$.

Definition 2.2. A function $F(x_1, \ldots, x_k)$ is said to be isovertible if it is an invertible function in each of its independent variables, i.e.,

$$F_1(\lambda) = F(\lambda, x_2, \ldots, x_k), \quad F_2(\lambda) = F(x_1, \lambda, x_3, \ldots, x_k), \ldots, F_k(\lambda) = F(x_1, \ldots, x_{k-1}, \lambda)$$

are invertible functions. The inverse of F_j is denoted by $F^{-(j)}$, i.e., $F^{-(j)} = F_j^{-1}$.

Remark 2.1. Observe that if $g, h : D \to D$ are two invertible functions, and $F : D^k \to D$ is symmetric and isovertible, then so the function G defined by

$$G(x_1, \ldots, x_k) = g(F(h(x_1), \ldots, h(x_k))).$$

We introduce in the following definition a class of functions that will play an important role in what follows.

Definition 2.3. Given a nonnegative integer r,

$$I_r(x_1, \ldots, x_k) = \begin{cases} 1 & \text{if } r = 0 \\ \displaystyle\sum_{1 \le i_1 < \cdots < i_r \le k} \prod_{s=1}^{r} x_{i_s} & \text{if } 0 < r \le k \\ 0 & \text{if } r > k. \end{cases}$$

For example, for $k = 2$ we have $I_1 = x_1 + x_2$, $I_2 = x_1 x_2$, for $k = 3$ we have $I_1 = x_1 + x_2 + x_3$, $I_2 = x_1 x_2 + x_1 x_3 + x_2 x_3$, $I_3 = x_1 x_2 x_3$, and so on.

Lemma 2.1. *Let r be a nonnegative integer such that $0 < r \le k$ and I_r be as in Definition 2.3, then*

(a) *I_r is symmetric and isovertible.*
(b) *For $j = 1, \ldots, k$, $I_r^{-(j)}$ is given by*

$$I_r^{-(j)}(x_1, \ldots, x_{j-1}, \lambda, x_{j+1}, \ldots, x_k) = \frac{\lambda - I_r(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k)}{I_{r-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k)}.$$

Proof. Part (a) is straightforward, and Part (b) follows from the following argument. First

$$\lambda = I_r(x_1, \ldots, x_j, \ldots, x_k) = \sum_{1 \le i_1 < \cdots < i_r \le k} \prod_{s=1}^{r} x_{i_s}$$

$$= \sum_{\substack{1\le i_1<\cdots<i_r\le k\\ i_s\neq j}} \prod_{s=1}^{r} x_{i_s} + x_j \sum_{\substack{1\le i_1<\cdots<i_r\le k\\ i_s\neq j}} \prod_{s=1}^{r-1} x_{i_s}$$
$$= I_r(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_k) + x_j I_{r-1}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_k).$$

Now, the result follows by solving for x_j. This completes the proof.

3. The main results

Theorem 3.1 below is our main result in this paper. It establishes, under restriction on the desired period p, a constructive method that generates one-parameter families of functions for which all solutions of Eq. (1) are periodic of period p.

Theorem 3.1. *Let p be a positive integer such that $p > k$. Suppose that $p - k$ divides k with $\ell = k/(p-k)$. Then all solutions of Eq. (1) are periodic of period p if*

$$f(x_1,\ldots,x_k) = F^{-(\ell+1)}(x_1, x_{1+(p-k)},\ldots,x_{1+(\ell-1)(p-k)},\lambda)$$

for some symmetric and isovertible function $F : D^{\ell+1} \to D$ and arbitrary constant $\lambda \in D$.

Proof. By Eq. (1) we have

$$y_{n+k} = f(y_n,\ldots,y_{n+k-1}) = F^{-(\ell+1)}(y_n, y_{n+(p-k)},\ldots,y_{n+(\ell-1)(p-k)},\lambda).$$

Therefore, by invertibility in the $(\ell+1)$st independent variable and the fact that $k = \ell(p-k)$, we have

$$F(y_n, y_{n+(p-k)},\ldots,y_{n+(\ell-1)(p-k)}, y_{n+\ell(p-k)}) = \lambda \quad \text{for all } n \ge 0.$$

In particular for $n + p - k$, we have

$$F(y_{n+(p-k)}, y_{n+2(p-k)},\ldots,y_{n+\ell(p-k)}, y_{n+(\ell+1)(p-k)}) = \lambda.$$

But, by symmetry, we have

$$F(y_{n+(\ell+1)(p-k)}, y_{n+(p-k)}, y_{n+2(p-k)},\ldots,y_{n+\ell(p-k)}) = \lambda.$$

Hence, since F is also invertible in the first independent variable, we get

$$y_n = y_{n+(\ell+1)(p-k)} = y_{n+\ell(p-k)+p-k} = y_{n+k+p-k} = y_{n+p}.$$

This completes the proof.

Using Lemma 2.1 part (b) and Theorem 3.1, we have the following result.

Corollary 3.1. *Suppose that k, p are positive integers such that $p > k \ge 2$ and $\ell = k/(p-k)$ is a positive integer. Then all solutions of Eq. (1) are periodic of period p if*

$$f(x_1,\ldots,x_k) = \frac{\lambda - I_r(x_1, x_{(p-k)},\ldots,x_{(\ell-1)(p-k)})}{I_{r-1}(x_1, x_{(p-k)},\ldots,x_{(\ell-1)(p-k)})}, \quad r = 1,2,\ldots,\frac{p}{p-k}$$

where λ is an arbitrary real number.

4. Examples

To illustrate the applicability of Theorem 3.1 and Corollary 3.1, we present the following examples.

Example 4.1. Suppose that $k = 2$. In this case there are only two possibilities for p, namely $p = 3$ or 4. On the one hand, if $p = 3$, then $\ell = 2$ and so all solutions of the difference equations

$$y_{n+2} = \frac{\lambda - I_r(y_n, y_{n+1})}{I_{r-1}(y_n, y_{n+1})}, \quad r = 1, 2, 3$$

are periodic of period 3. In this case, we have the difference equations:

$$\begin{aligned}
r = 1: &\quad y_{n+2} = \frac{\lambda - I_1(y_n, y_{n+1})}{I_0(y_n, y_{n+1})} = \lambda - y_n - y_{n+1} \\
r = 2: &\quad y_{n+2} = \frac{\lambda - I_2(y_n, y_{n+1})}{I_1(y_n, y_{n+1})} = \frac{\lambda - y_n y_{n+1}}{y_n + y_{n+1}} \\
r = 3: &\quad y_{n+2} = \frac{\lambda - I_3(y_n, y_{n+1})}{I_2(y_n, y_{n+1})} = \frac{\lambda}{y_n y_{n+1}}.
\end{aligned}$$

On the other hand, if $p = 4$, then $\ell = 1$ and so all solutions of the difference equations

$$y_{n+2} = \frac{\lambda - I_r(y_n)}{I_{r-1}(y_n)}, \quad r = 1, 2$$

are periodic of period 4. In this case, we have the difference equations:

$$\begin{aligned}
r = 1: &\quad y_{n+2} = \frac{\lambda - I_1(y_n)}{I_0(y_n)} = \lambda - y_n \\
r = 2: &\quad y_{n+2} = \frac{\lambda - I_2(y_n)}{I_1(y_n)} = \frac{\lambda}{y_n}.
\end{aligned}$$

Example 4.2. Suppose that $k = 3$. In this case, again, there are only two possibilities for p, namely $p = 4$ or 6. On the one hand, if $p = 4$, then $\ell = 3$ and so all solutions of the difference equations

$$y_{n+3} = \frac{\lambda - I_r(y_n, y_{n+1}, y_{n+2})}{I_{r-1}(y_n, y_{n+1}, y_{n+2})}, \quad r = 1, 2, 3, 4$$

are periodic of period 4. In this case, we have the following difference equations:

$$\begin{aligned}
r = 1: &\quad y_{n+3} = \frac{\lambda - I_1(y_n, y_{n+1}, y_{n+2})}{I_0(y_n, y_{n+1}, y_{n+2})} = \lambda - y_n - y_{n+1} - y_{n+2} \\
r = 2: &\quad y_{n+2} = \frac{\lambda - I_2(y_n, y_{n+1}, y_{n+2})}{I_1(y_n, y_{n+1}, y_{n+3})} = \frac{\lambda - y_n y_{n+1} - y_n y_{n+1} - y_{n+1} y_{n+2}}{y_n + y_{n+1} + y_{n+2}} \\
r = 3: &\quad y_{n+2} = \frac{\lambda - I_3(y_n, y_{n+1}, y_{n+2})}{I_2(y_n, y_{n+1}, y_{n+3})} = \frac{\lambda - y_n y_{n+1} y_{n+2}}{y_n y_{n+1} + y_n y_{n+2} + y_{n+1} y_{n+2}} \\
r = 4: &\quad y_{n+3} = \frac{\lambda - I_4(y_n, y_{n+1}, y_{n+2})}{I_3(y_n, y_{n+1}, y_{n+2})} = \frac{\lambda}{y_n y_{n+1} y_{n+2}}.
\end{aligned}$$

On the other hand, if $p = 6$, then $\ell = 1$ and so all solutions of the difference equations

$$y_{n+3} = \frac{\lambda - I_r(y_n)}{I_{r-1}(y_n)}, \quad r = 1, 2$$

are periodic of period 6. In this case, we have the difference equations:

$$r = 1: \quad y_{n+3} = \frac{\lambda - I_1(y_n)}{I_0(y_n)} = \lambda - y_n$$
$$r = 2: \quad y_{n+3} = \frac{\lambda - I_2(y_n)}{I_1(y_n)} = \frac{\lambda}{y_n}.$$

5. Conclusion

In this note, we developed a constructive method that generates one-parameter families of functions f of k independent variables such that all solutions of Eq. (1) are periodic of the same period p. To achieve this goal we assumed that $(p - k)$ divides k. We wonder if this restriction can be removed. For definiteness, we ask the following question:

Given positive integers p, k *such that* $p \geq k \geq 2$, *find, if possible, a function* f *of* k *independent variables such that every solution of* (1) *is periodic of period* p.

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